Wednesday, October 28, 2015

539: 1, 2, 4, 5, 7, 9, 11, 13, 15, 55, 56

Problem 1

Problem. State the trigonometric substitution you would use to find the indefinite integral $\int (9 + x^2)^{-2}\, dx$.

Solution. I would let $x = 3\tan\theta$, $dx = 3\sec^2\theta\; d\theta$, and $\sqrt{9 + x^2} = 3\sec\theta$.

Problem 2

Problem. State the trigonometric substitution you would use to find the indefinite integral $\int \sqrt{4 - x^2}\, dx$.

Solution. I would let $x = 2\sin\theta$, $dx = 2\cos\theta\; d\theta$, and $\sqrt{4 - x^2} = 2\cos\theta$.

Problem 4

Problem. State the trigonometric substitution you would use to find the indefinite integral $\int x^2(x^2 - 25)^{3/2}\, dx$.

Solution. I would let $x = 5\sec\theta$, $dx = 5\tan\theta\sec\theta\; d\theta$, and $\sqrt{x^2 - 25} = 5\tan\theta$.

Problem 5

Problem. Find the indefinite integral $\int \frac{1}{(16 - x^2)^{3/2}}\, dx$ using the substitution $x = 4\sin\theta$.

Solution. Let $x = 4\sin\theta$, $dx = 4\cos\theta\, d\theta$, and $\sqrt{16 - x^2} = 4\cos\theta$. Then

$$\begin{aligned}\int \frac{1}{(16-x^2)^{3/2}}\, dx &= \int \frac{1}{(\sqrt{16-x^2})^3}\, dx \\ &= \int \frac{1}{(4\cos\theta)^3} \cdot 4\cos\theta\, d\theta \\ &= \int \frac{1}{(4\cos\theta)^2}\, d\theta \\ &= \frac{1}{16}\int \sec^2\theta\, d\theta \\ &= \frac{1}{16}\tan\theta + C \\ &= \frac{1}{16} \cdot \frac{x}{\sqrt{16-x^2}} + C \\ &= \frac{x}{16\sqrt{16-x^2}} + C.\end{aligned}$$

Problem 7

Problem. Find the indefinite integral $\int \frac{\sqrt{16-x^2}}{x}\, dx$ using the substitution $x = 4\sin\theta$.

Solution. Let $x = 4\sin\theta$, $dx = 4\cos\theta\, d\theta$, and $\sqrt{16 - x^2} = 4\cos\theta$. Then

$$\begin{aligned}\int \frac{\sqrt{16-x^2}}{x}\, dx &= \int \frac{4\cos\theta}{4\sin\theta} \cdot 4\cos\theta\, d\theta \\ &= 4\int \frac{\cos^2\theta}{\sin\theta}\, d\theta \\ &= 4\int \left(\frac{1-\sin^2 x}{\sin x}\right)\, dx \\ &= 4\int \csc x\, dx - 4\int \sin x\, dx \\ &= -4\ln|\cot x + \csc x| + 4\cos x + C \\ &= -4\ln|\frac{\sqrt{16-x^2}}{x} + \frac{4}{x}| + 4 \cdot \frac{\sqrt{16-x^2}}{4} + C \\ &= -4\ln\left|\frac{4+\sqrt{16-x^2}}{x}\right| + \sqrt{16-x^2} + C.\end{aligned}$$

Problem 9

Problem. Find the indefinite integral $\int \frac{1}{\sqrt{x^2-25}}\,dx$ using the substitution $x = 5\sec\theta$.

Solution. Let $x = 5\sec\theta$, $dx = 5\tan\theta\sec\theta\,d\theta$, and $\sqrt{x^2-25} = 5\tan\theta$. Then

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-25}}\,dx &= \int \frac{1}{5\tan\theta}\cdot 5\tan\theta\sec\theta\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\tan\theta + \sec\theta| + C \\
&= \ln\left|\frac{\sqrt{x^2-25}}{5} + \frac{x}{5}\right| + C \\
&= \ln\left|\frac{x+\sqrt{x^2-25}}{5}\right| + C.
\end{aligned}$$

Problem 11

Problem. Find the indefinite integral $\int x^3\sqrt{x^2-25}\,dx$ using the substitution $x = 5\sec\theta$.

Solution. Let $x = 5\sec\theta$, $dx = 5\tan\theta\sec\theta\,d\theta$, and $\sqrt{x^2-25} = 5\tan\theta$. Then

$$\begin{aligned}
\int x^3\sqrt{x^2-25}\,dx &= \int (5\sec\theta)^3\cdot 5\tan\theta\cdot 5\tan\theta\sec\theta\,d\theta \\
&= 3125\int \tan^2\theta\sec^4\theta\,d\theta \\
&= 3125\int \tan^2\theta(1+\tan^2\theta)\sec^2\theta\,d\theta
\end{aligned}$$

Let $u = \tan\theta$ and $du = \sec^2\theta\, d\theta$. Then

$$\begin{aligned}
3125\int \tan^2\theta(1+\tan^2\theta)\sec^2\theta\, d\theta &= 3125\int u^2(1+u^2)\, du \\
&= 3125\int (u^2+u^4)\, du \\
&= 3125\left(\frac{1}{3}u^3+\frac{1}{5}u^5\right)+C \\
&= 3125\left(\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta\right)+C \\
&= 3125\left(\frac{1}{3}\left(\frac{\sqrt{x^2-25}}{5}\right)^3+\frac{1}{5}\left(\frac{\sqrt{x^2-25}}{5}\right)^5\right)+C \\
&= 3125\left(\frac{(x^2-25)^{3/2}}{375}+\frac{(x^2-25)^{5/2}}{15625}\right)+C \\
&= \frac{25}{3}(x^2-25)^{3/2}+\frac{1}{5}(x^2-25)^{5/2}+C.
\end{aligned}$$

Problem 13

Problem. Find the indefinite integral $\int x\sqrt{1+x^2}\, dx$ using the substitution $x = \tan\theta$.

Solution. Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, and $\sqrt{1+x^2} = \sec\theta$. Then

$$\int x\sqrt{1+x^2}\, dx = \int \tan\theta\cdot\sec\theta\cdot\sec^2\theta\, d\theta.$$

Let $u = \sec\theta$ and $du = \tan\theta\sec\theta\, d\theta$. Then

$$\begin{aligned}
\int \tan\theta\cdot\sec\theta\cdot\sec^2\theta\, d\theta &= \int u^2\, du \\
&= \frac{1}{3}u^3+C \\
&= \frac{1}{3}\sec^3\theta+C \\
&= \frac{1}{3}(\sqrt{1+x^2})^3\theta+C \\
&= \frac{1}{3}(1+x^2)^{3/2}+C
\end{aligned}$$

Note that this problem could have been solved easily by the substitution $u = 1+x^2$.

Problem 15

Problem. Find the indefinite integral $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$ using the substitution $x = \tan\theta$.

Solution. Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, and $\sqrt{1+x^2} = \sec\theta$. Then

$$
\begin{aligned}
\int \frac{1}{(1+x^2)^2}\,dx &= \int \frac{1}{\sec^4\theta}\cdot\sec^2\theta\,d\theta \\
&= \int \frac{1}{\sec^2\theta}\,d\theta \\
&= \int \cos^2\theta\,d\theta \\
&= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C \\
&= \frac{1}{2}\arctan x + \frac{1}{2}\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) + C \\
&= \frac{1}{2}\arctan x + \frac{x}{2(1+x^2)} + C
\end{aligned}
$$

Problem 55

Problem. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution. We will find the area of the upper half and double it. The equation of the upper half is

$$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}.$$

So the area is

$$
\begin{aligned}
2\int_a^a b\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx &= 2b\int_a^a \sqrt{1-\left(\frac{x}{a}\right)^2}\,dx \\
&= \frac{2b}{a}\int_a^a \sqrt{a^2-x^2}\,dx.
\end{aligned}
$$

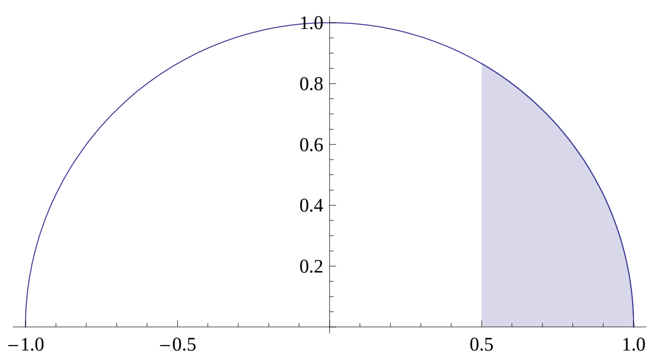
Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$, and $\sqrt{a^2 - x^2} = a\cos\theta$. Then

$$\begin{aligned}
\frac{2b}{a}\int_a^a \sqrt{a^2 - x^2}\, dx &= \frac{2b}{a}\int_{-\pi/2}^{\pi/2} a\cos\theta \cdot a\cos\theta\, d\theta \\
&= 2ab\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta \\
&= ab\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta\, d\theta) \\
&= ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} \\
&= ab\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\
&= \pi ab.
\end{aligned}$$

Problem 56

Problem. Find the area of the shaded region of the circle of radius a when the chord is h units from the center of the circle.

Solution. It will be simpler if we rotate the diagram 90° clockwise and then find the area of the upper half and double it.



$$\text{Area} = 2\int_h^a \sqrt{a^2 - x^2}\, dx.$$

Let $x = a \sin\theta$, $dx = a\cos\theta \, d\theta$, and $\sqrt{a^2 - x^2} = a\cos\theta$. Then

$$\begin{aligned}
2\int_h^a \sqrt{a^2 - x^2}\, dx &= 2a^2 \int_{\arcsin h/a}^{\pi/2} \cos^2\theta \, d\theta \\
&= a^2 \int_{\arcsin h/a}^{\pi/2} (1 + \cos 2\theta)\, d\theta \\
&= a^2 \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\arcsin h/a}^{\pi/2} \\
&= a^2 \left[\theta + \sin\theta\cos\theta\right]_{\arcsin h/a}^{\pi/2} \\
&= a^2 \left(\frac{\pi}{2} - \left(\arcsin\frac{h}{a} + \frac{h}{a}\cdot\frac{\sqrt{a^2 - h^2}}{a}\right)\right) \\
&= \frac{\pi a^2}{2} - a^2 \arcsin\frac{h}{a} - h\sqrt{a^2 - h^2}.
\end{aligned}$$